

TWO-SIDED MULTIPLICATION OPERATORS ON THE SPACE OF REGULAR OPERATORS

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ABSTRACT. Let W , X , Y and Z be Dedekind complete Riesz spaces. For $A \in L^r(Y, Z)$ and $B \in L^r(W, X)$ let $M_{A, B}$ be the two-sided multiplication operator from $L^r(X, Y)$ into $L^r(W, Z)$ defined by $M_{A, B}(T) = ATB$. We show that for every $0 \leq A_0 \in L_n^r(Y, Z)$, $|M_{A_0, B}|(T) = M_{A_0, |B|}(T)$ holds for all $B \in L^r(W, X)$ and all $T \in L_n^r(X, Y)$. Furthermore, if W , X , Y and Z are Dedekind complete Banach lattices such that X and Y have order continuous norms, then $|M_{A, B}| = M_{|A|, |B|}$ for all $A \in L^r(Y, Z)$ and all $B \in L^r(W, X)$. Our results generalize the related results of Synnatzschke and Wickstead, respectively.

1. INTRODUCTION

For an algebra \mathcal{A} an operator of the form $T \mapsto \sum_{i=1}^n A_i T B_i$, where A_i, B_i are fixed in \mathcal{A} , is referred to as an *elementary operator* on \mathcal{A} . If $A, B \in \mathcal{A}$, we denote by $M_{A, B}$ the operator $T \mapsto ATB$. The operator $M_{A, B}$ is called a *basic elementary operator* or a *two-sided multiplication operator*. The literature related to (basic) elementary operators is by now very large, much of it in the setting of C^* -algebras or in the Banach space setting. In this direction there are many excellent surveys and expositions. See, e.g., [4, 5, 6, 8].

For the study of two-sided multiplication operators in the setting of Riesz spaces (i.e., vector lattices) we would like to mention the work of Synnatzschke [9]. The set of all regular operators (order continuous regular operators, resp.) from a Riesz space X into a Dedekind complete Riesz space Y will be denoted by $L^r(X, Y)$ ($L_n^r(X, Y)$, resp.). When $Y = \mathbb{R}$, we write X^\sim and X_n^\sim respectively instead of $L^r(X, \mathbb{R})$ and $L_n^r(X, \mathbb{R})$. They are likewise Dedekind complete Riesz spaces. Let W, X, Y and Z be Dedekind complete Riesz spaces. For all $A \in L^r(Y, Z)$ and $B \in L^r(W, X)$, $M_{A, B} : T \in L^r(X, Y) \mapsto ATB \in L^r(W, Z)$ is a regular operator, and hence the modulus $|M_{A, B}|$ of $M_{A, B}$ exists in $L^r(L^r(X, Y), L^r(W, Z))$. It is interesting to know about the relationship of $|M_{A, B}|$ with $|A|$ and $|B|$. Among other things, Synnatzschke [9, Satz 3.1] proved the following result:

a) If $0 \leq B_0 \in L^r(W, X)$, then $|M_{A, B_0}| = M_{|A|, B_0}$, $M_{A, B_0} \vee M_{C, B_0} = M_{A \vee C, B_0}$ hold for all $A, C \in L^r(Y, Z)$.

b) If $0 \leq A_0 \in L_n^r(Y, Z)$ and Y_n^\sim, Z_n^\sim are total, then we have $|M_{A_0, B}|(T) = M_{A_0, |B|}(T)$ and $(M_{A_0, B} \vee M_{A_0, D})(T) = M_{A_0, B \vee D}(T)$ for all $B, D \in L^r(W, X)$ and all $T \in L_n^r(X, Y)$.

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Hereby Y_n^\sim is *total* if Y_n^\sim separates the points of Y . Synnatzschke used (a) to establish (b) by taking adjoints of operators. For his purpose, the hypothesis of both Y_n^\sim and Z_n^\sim being total is essential.

Recently, Wickstead [10] showed that if E is an atomic Banach lattice with order continuous norm and $A, B \in L^r(E)$, then $|M_{A,B}| = M_{|A|,|B|}$ and $\|M_{A,B}\|_r = \|A\|_r \|B\|_r$. In his proofs he depended heavily upon the ‘atomic’ condition.

In this note, we generalize the related results of Synnatzschke and Wickstead, respectively. We remove the condition of order continuous duals being total in [9, Satz 3.1(b)] and show that for every $0 \leq A_0 \in L_n^r(Y, Z)$, $|M_{A_0,B}|(T) = M_{A_0,|B|}(T)$ holds for all $B \in L^r(W, X)$ and all $T \in L_n^r(X, Y)$. Furthermore, if W, X, Y and Z are Dedekind complete Banach lattices such that X and Y have order continuous norms (not necessarily atomic), then $|M_{A,B}| = M_{|A|,|B|}$ and $\|M_{A,B}\|_r = \|A\|_r \|B\|_r$ hold for all $A \in L^r(Y, Z)$ and all $B \in L^r(W, X)$.

Our notions are standard. For the theory of Riesz spaces and regular operators, we refer the reader to the monographs [2, 7, 11].

2. THE MODULUS OF THE TWO-SIDED MULTIPLICATION OPERATOR

We start with two examples which serve to illustrate that the order continuous dual X_n^\sim of a Dedekind complete Riesz space X need not be total. This justifies our effort to generalize the result of Synnatzschke [9, Satz 3.1 b)].

Example 2.1. (1) Let (Ω, Σ, μ) be a nonatomic finite measure space. Then the Dedekind complete Riesz space $X = L_p(\Omega, \Sigma, \mu)$ ($0 < p < 1$) satisfies $X_n^\sim = X^\sim = \{0\}$. This result is due to M. M. Day. (cf. [3, Theorem 5.24, p. 128]).

(2) Let K be a compact Hausdorff space. It is well known that $C(K)$ is Dedekind complete if and only if K is Stonian (i.e., extremally disconnected). A Hausdorff compact Stonian space K such that $C(K)_n^\sim$ is total is called hyper-Stonian. Dixmier gave a characterization of hyper-Stonian spaces: K is hyper-Stonian if and only if $C(K)$ is isomorphic to a dual Banach lattice (cf. [7, Theorem 2.1.7]). He also gave a Dedekind complete $C(K)$ -space which is not isomorphic to a dual space (see, e.g., [1, p. 99, Problems 4.8 and 4.9] for details). That is, such a $C(K)$ is Dedekind complete, but $C(K)_n^\sim$ is not total.

Proposition 2.2. *Let W, X, Y and Z be Riesz spaces with X, Y and Z Dedekind complete. Let $0 \leq A_0 \in L_n^r(Y, Z)$. Then we have $|M_{A_0,B}|(T) = M_{A_0,|B|}(T)$ and, equivalently, $M_{A_0,B} \vee M_{A_0,D}(T) = M_{A_0,B \vee D}(T)$ for all $B, D \in L^r(W, X)$ and all $T \in L_n^r(X, Y)$.*

Proof. For $B \in L^r(W, X)$ and $0 \leq T \in L_n^r(X, Y)$, we have to prove that $|M_{A_0,B}|(T) = M_{A_0,|B|}(T)$. Clearly we have $|M_{A_0,B}|(T) \leq M_{A_0,|B|}(T)$, since $|M_{A_0,B}| \leq M_{A_0,|B|}$ holds in $L^r(L^r(X, Y), L^r(W, Z))$. For the reverse inequality, let $w \in W^+$. By a formula for the modulus of regular operators in [2, Theorem 1.21(3)] or [11, Theorem 20.10(i)] we have

$$\left(\sum_{i=1}^n |Bw_i| : n \in \mathbb{N}, 0 \leq w_i \in W, \sum_i w_i = w \right) \uparrow |B|w.$$

Since A_0 and T are both positive order continuous operators, A_0T is likewise an order continuous positive operator from X into Z . It follows that

$$\begin{aligned} M_{A_0, |B|}(T)(w) &= A_0T|B|w \\ &= \sup \left(\sum_{i=1}^n A_0T|Bw_i| : n \in \mathbb{N}, 0 \leq w_i \in W, \sum_i w_i = w \right) \end{aligned}$$

For each $1 \leq i \leq n$, let P_i be the order projection from X onto the band generated by $(Bw_i)^+$ in X and let $Q_i = P_i - I$, where I is the identity operator on X . Clearly,

$$P_i \perp Q_i, \quad |P_i| + |Q_i| = I, \quad P_iBw_i = (Bw_i)^+, \quad (P_i + Q_i)Bw_i = |Bw_i|,$$

and

$$|TP_i| + |TQ_i| = T.$$

Therefore, for each i we have

$$\begin{aligned} A_0T|Bw_i| &= (A_0TP_i + A_0TQ_i)Bw_i \\ &\leq (|A_0(TP_i)B| + |A_0(TQ_i)B|)w_i \\ &\leq \left(\sup \left\{ \sum_{j=1}^m |A_0T_jB| : m \in \mathbb{N}, T_j \in L^r(X, Y), \sum_j |T_j| = T \right\} \right) w_i \\ &= \left(\sup \left\{ \sum_{j=1}^m |M_{A_0, B}(T_j)| : m \in \mathbb{N}, T_j \in L^r(X, Y), \sum_j |T_j| = T \right\} \right) w_i \\ &= |M_{A_0, B}|(T)(w_i). \end{aligned}$$

Hence, from this it follows that

$$\begin{aligned} M_{A_0, |B|}(T)(w) &= \sup \left(\sum_{i=1}^n A_0T|Bw_i| : n \in \mathbb{N}, 0 \leq w_i \in W, \sum_i w_i = w \right) \\ &\leq \sup \left(\sum_{i=1}^n |M_{A_0, B}|(T)(w_i) : n \in \mathbb{N}, 0 \leq w_i \in W, \sum_i w_i = w \right) \\ &= |M_{A_0, B}|(T)(w), \end{aligned}$$

which implies that $|M_{A_0, B}|(T) \leq M_{A_0, |B|}(T)$ for all $B \in L^r(W, X)$ and all $0 \leq T \in L_n^r(X, Y)$. \square

In general we can not expect that $|M_{A_0, B}| = M_{A_0, |B|}$ holds for all $B \in L^r(W, X)$. That is, the linear operator $M_{A_0, \cdot} : B \in L^r(W, X) \rightarrow L^r(L^r(X, Y), L^r(W, Z))$ is not necessarily a Riesz homomorphism. In the last section we give a counterexample to illustrate this. However, for Banach lattices with order continuous norms the situation is quite different. The next result is a consequence of the above proposition and the earlier result of Synnatzschke [9, Satz 3.1], which generalizes Theorem 3.1 of Wickstead [10] recently obtained for atomic Banach lattices with order continuous norms.

Corollary 2.3. *Let W, X, Y and Z be Banach lattices such that X, Y have order continuous norms and Z is Dedekind complete. Then we have $|M_{A, B}| = M_{|A|, |B|}$ for all $A \in L^r(Y, Z)$ and all $B \in L^r(W, X)$.*

Proof. Let $\mathcal{M} : L^r(Y, Z) \times L^r(W, X) \rightarrow L^r(L^r(X, Y), L^r(W, Z))$ be the bilinear operator defined via $\mathcal{M}(A, B) = M_{A, B}$. Clearly, \mathcal{M} is positive. Since X and Y are Banach lattices with order continuous norms, we have $L_n^r(X, Y) = L^r(X, Y)$ and $L_n^r(Y, Z) = L^r(Y, Z)$. From Proposition 2.2 above and the result of Synnatzschke [9, Satz 3.1] it follows that for every $0 \leq A_0 \in L^r(Y, Z)$ and every $0 \leq B_0 \in L^r(W, X)$, $\mathcal{M}(A_0, \cdot)$ and $\mathcal{M}(\cdot, B_0)$ are both Riesz homomorphisms. Hence, for all $A \in L^r(Y, Z)$ and all $B \in L^r(W, X)$ we have

$$\begin{aligned} |M_{A, B}| &= |\mathcal{M}(A, B)| \\ &= |\mathcal{M}(A^+ - A^-, B^+ - B^-)| \\ &= |\mathcal{M}(A^+, B^+) - \mathcal{M}(A^+, B^-) - \mathcal{M}(A^-, B^+) + \mathcal{M}(A^-, B^-)| \\ &= \mathcal{M}(A^+, B^+) + \mathcal{M}(A^+, B^-) + \mathcal{M}(A^-, B^+) + \mathcal{M}(A^-, B^-) \\ &= \mathcal{M}(|A|, |B|) = M_{|A|, |B|}. \end{aligned}$$

Here we are using the fact that the terms $\mathcal{M}(A^+, B^+)$, $\mathcal{M}(A^+, B^-)$, $\mathcal{M}(A^-, B^+)$ and $\mathcal{M}(A^-, B^-)$ are pairwise disjoint. \square

Let W and X be Banach lattices with X Dedekind complete. Recall that $L^r(W, X)$ is a Dedekind complete Banach lattice under the regular norm $\|B\|_r := \|\|B\|\|$ for every $B \in L^r(W, X)$. Note that $M_{A, B}$ is a regular operator from $L^r(X, Y)$ into $L^r(W, Z)$. The following result deals with the regular norms of two-sided multiplication operators. Its proof is based on Corollary 2.3

Corollary 2.4. *If W, X, Y and Z be Banach lattices such that X, Y have order continuous norms and Z is Dedekind complete, then $\|M_{A, B}\|_r = \|A\|_r \|B\|_r$ for all $A \in L^r(Y, Z)$ and all $B \in L^r(W, X)$.*

Proof. We first assume that $0 \leq A \in L^r(Y, Z)$ and $0 \leq B \in L^r(W, X)$. Since $M_{A, B} \geq 0$, we have $\|M_{A, B}\|_r = \|M_{A, B}\| \leq \|A\| \|B\| = \|A\|_r \|B\|_r$. On the other hand, for every $0 \leq x' \in X'$ and every $0 \leq y \in Y$ satisfying $\|x'\| \leq 1$ and $\|y\| \leq 1$, $x' \otimes y \in L^r(X, Y)$ and $\|x' \otimes y\|_r = \|x' \otimes y\| \leq 1$. Then it follows that

$$\begin{aligned} \|M_{A, B}\|_r = \|M_{A, B}\| &\geq \sup \left(\|M_{A, B}(x' \otimes y)\| : 0 \leq x' \in B_{X'}, 0 \leq y \in B_Y \right) \\ &= \sup \left(\|(B'x') \otimes Ay\| : 0 \leq x' \in B_{X'}, 0 \leq y \in B_Y \right) \\ &= \|A\| \|B\| = \|A\|_r \|B\|_r. \end{aligned}$$

This implies that $\|M_{A, B}\|_r = \|A\|_r \|B\|_r$ holds for all $0 \leq A \in L^r(Y, Z)$ and $0 \leq B \in L^r(W, X)$.

Now, for the general case let $A \in L^r(Y, Z)$ and $B \in L^r(W, X)$ be arbitrary. Then by Corollary 2.3 we have

$$\|M_{A, B}\|_r = \|\|M_{A, B}\|\| = \|M_{|A|, |B|}\| = \|M_{|A|, |B|}\|_r = \|A\|_r \|B\|_r.$$

\square

Wickstead [10] establishes that even in the case of atomic Banach lattices with order continuous norms the operator norm of two-sided multiplication operators need not be equivalent to the regular norm.

3. A COUNTEREXAMPLE

Let X and Y be Riesz spaces with Y Dedekind complete. The set of all σ -order continuous operators in $L^r(X, Y)$ will be denoted by $L_c^r(X, Y)$. The disjoint complement $(L_c^r(X, Y))^d$ of $L_c^r(X, Y)$ is denoted by $L_s^r(X, Y)$. Every element of $L_s^r(X, Y)$ is called a singular operator. When $Y = \mathbb{R}$, we write X^\sim and X_s^\sim respectively instead of $L^r(X, \mathbb{R})$ and $L_s^r(X, \mathbb{R})$. The following example illustrates that $|M_{A_0, B}| = M_{A_0, |B|}$ does not necessarily hold for all $B \in L^r(W, X)$, that is, the linear operator $M_{A_0, \cdot} : B \in L^r(W, X) \rightarrow L^r(L^r(X, Y), L^r(W, Z))$ is not necessarily a Riesz homomorphism in general.

Example 3.1. Let $W = X = Y = Z = \ell_\infty$ and let e denote the strong unit $(1, 1, \dots)$ of ℓ_∞ . Let $0 \leq f \in (\ell_\infty)_s^\sim$ be a singular Riesz homomorphism with $f(e) = 1$ (one can take, e.g., f equal to a limit over a free ultrafilter). Let $B \in L^r(\ell_\infty)$ be the rank one operator $B = f \otimes e$. Then it is clear that $B \in L_s^r(\ell_\infty)$ and $I \wedge B = 0$, where I is the identity operator on ℓ_∞ (and hence order continuous). We claim that $M_{I, I} \wedge M_{I, B} \neq M_{I, I \wedge B} = 0$. To this end, let $0 \leq T \in L^r(\ell_\infty)$. Then, by [2, Theorem 1.21(2)] we have

$$\left\{ \sum_{i=1}^n (T_i \wedge T_i B) : n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\} \downarrow (M_{I, I} \wedge M_{I, B})(T).$$

From this and [2, Theorem 1.51(2)] it follows that

$$\begin{aligned} & (M_{I, I} \wedge M_{I, B})(T)(e) \\ &= \inf \left\{ \sum_{i=1}^n (T_i \wedge T_i B)(e) : n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\} \\ &= \inf \left\{ \sum_{i=1}^n (T_i \wedge (f \otimes T_i e))(e) : n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\} \\ &= \inf \left\{ \sum_{i=1}^n \inf \left(\sum_{j=1}^{m_i} T_i x_j^{(i)} \wedge f(x_j^{(i)}) T_i e : x_j^{(i)} \wedge x_k^{(i)} = 0, j \neq k, \sum_{j=1}^{m_i} x_j^{(i)} = e \right) : \right. \\ & \quad \left. n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\}. \end{aligned}$$

Let us put

$$\begin{aligned} G' &= \left\{ \sum_{i=1}^n \inf \left(\sum_{j=1}^{m_i} T_i x_j^{(i)} \wedge f(x_j^{(i)}) T_i e : x_j^{(i)} \wedge x_k^{(i)} = 0, j \neq k, \sum_{j=1}^{m_i} x_j^{(i)} = e \right) : \right. \\ & \quad \left. n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\}, \\ G'' &= \left\{ \sum_{i=1}^n \sum_{j=1}^m T_i x_j \wedge f(x_j) T_i e : m \in \mathbb{N}, x_j \wedge x_k = 0, j \neq k, \sum_{j=1}^m x_j = e, \right. \\ & \quad \left. n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\}. \end{aligned}$$

We claim that $\inf G' = \inf G''$. Indeed, it is clear that $\inf G' \leq \inf G''$. For the reverse inequality, let $(T_i)_1^n$ be a fixed positive partition of T (i.e., $T_i \geq 0$, $\sum_i T_i = T$). For each i let $(x_j^{(i)})_{j=1}^{m_i}$ be an arbitrary positive disjoint partition of e (i.e., $x_j^{(i)} \wedge x_k^{(i)} = 0, j \neq k, \sum_{j=1}^{m_i} x_j^{(i)} = e$) corresponding to T_i . Following the proof of [2, Theorem 1.51] we can find a positive disjoint partition $(x_j)_1^m$ of e such that

$$\sum_{j=1}^m T_i x_j \wedge f(x_j) T_i e \leq \sum_{j=1}^{m_i} T_i x_j^{(i)} \wedge f(x_j^{(i)}) T_i e \quad (i = 1, 2, \dots, n).$$

From this it follows that

$$\inf G'' \leq \sum_{i=1}^n \sum_{j=1}^m T_i x_j \wedge f(x_j) T_i e \leq \sum_{i=1}^n \sum_{j=1}^{m_i} T_i x_j^{(i)} \wedge f(x_j^{(i)}) T_i e$$

Therefore,

$$\inf G'' \leq \sum_{i=1}^n \inf \left(\sum_{j=1}^{m_i} T_i x_j^{(i)} \wedge f(x_j^{(i)}) T_i e : x_j^{(i)} \wedge x_k^{(i)} = 0, j \neq k, \sum_{j=1}^{m_i} x_j^{(i)} = e \right),$$

which implies that $\inf G'' \leq \inf G'$. Hence, we have $(M_{I,I} \wedge M_{I,B})(T)(e) = \inf G''$.

Since f is a Riesz homomorphism, for every positive disjoint partition $(x_j)_1^m$ of e appearing in G'' there exists only one, say x_{j_0} , such that

$$f(x_j) = 0, \quad j \neq j_0, \quad f(x_{j_0}) = f(e) = 1$$

$$x_{j_0} \wedge \sum_{j \neq j_0} x_j = x_{j_0} \wedge (e - x_{j_0}) = 0.$$

It follows that $\sum_{i=1}^n \sum_{j=1}^m T_i x_j \wedge f(x_j) T_i e = \sum_{i=1}^n T_i x_{j_0}$. On the other hand, for any $x \in E^+$ satisfying $x \wedge (e - x) = 0$ and $f(x) = 1$, we must have $f(e - x) = 0$, and hence

$$\sum_{i=1}^n T_i x = \sum_{i=1}^n \left(T_i x \wedge f(x) T_i e + T_i (e - x) \wedge f(e - x) T_i e \right)$$

Thus, we have

$$\begin{aligned} & (M_{I,I} \wedge M_{I,B})(T)(e) \\ &= \inf \left\{ \sum_{i=1}^n \sum_{j=1}^m T_i x_j \wedge f(x_j) T_i e : m \in \mathbb{N}, x_j \wedge x_k = 0, j \neq k, \sum_{j=1}^m x_j = e, \right. \\ & \quad \left. n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\} \\ &= \inf \left\{ \sum_{i=1}^n T_i x : x \wedge (e - x) = 0, f(x) = 1, n \in \mathbb{N}, T_i \geq 0, \sum_i T_i = T \right\} \\ &= \inf \left\{ T x : 0 \leq x \leq e, x \wedge (e - x) = 0, f(x) = 1 \right\}. \end{aligned}$$

If we now take $T = B = f \otimes e$, then $(M_{I,I} \wedge M_{I,B})(f \otimes e)(e) = e$. So, $M_{I,I} \wedge M_{I,B} \neq 0$.

REFERENCES

- [1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics **233**, Springer, New York, 2006. MR2006h:46005
- [2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators* (reprint of the 1985 original), Springer, Dordrecht, 2006. MR2262133
- [3] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, Math Surveys and Monographs, Volume #105, Amer. Math. Soc., 2003. MR2005b:46010.
- [4] P. Ara and M. Mathieu, *Local Multipliers of C^* -Algebras*, Springer Monographs in Mathematics, Springer-Verlag, London, 2003. MR2008i:47076
- [5] R. E. Curto, Spectral theory of elementary operators, *Elementary operators and applications* (M. Mathieu, ed.) (Proc. Int. Conf. Blaubeuren, 1991), 3-52, World Sci. Publ., River Edge, NJ, 1992. MR93i:47041
- [6] M. Mathieu, The norm problem for elementary operators, *Recent progress in functional analysis (Valencia, 2000)*, 363-368, North-Holland Math. Stud., **189**, North-Holland, Amsterdam, 2001. MR2002g:47071
- [7] P. Meyer-Nieberg, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991. MR 93f:46025
- [8] E. Saksman and H.-O. Tylli, Multiplications and elementary operators in the Banach space setting, *Methods in Banach space theory*, 253-292, London Math. Soc. Lecture Note Ser., **337**, Cambridge Univ. Press, Cambridge, 2006. MR2008i:47076
- [9] J. Synnatzschke, Über einige verbandstheoretische Eigenschaften der Multiplikation von Operatoren in Vektorverbänden, *Math. Nachr.* **95** (1980), 273-292. MR82b:47048
- [10] A. W. Wickstead, Norms of basic elementary operators on algebras of regular operators, *Proc. Amer. Math. Soc.*, to appear.
- [11] A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer, Berlin, 1997. MR2000c:47074

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